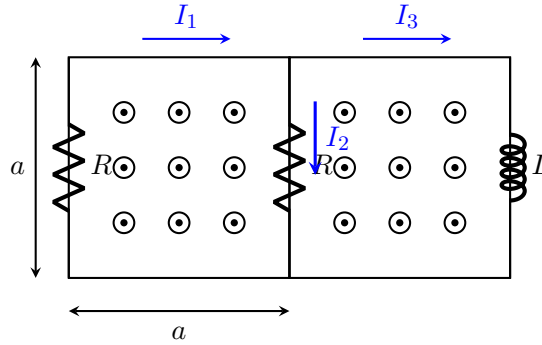


1. A circuit consists of two identical square loops of side length a that share a common central branch, as shown in the figure. The left branch contains a resistor R , the middle branch contains another resistor R , and the right branch contains an inductor L . A uniform magnetic field is applied perpendicular to the plane of the circuit (out of the page). It varies with time as $\vec{B}(t) = B_0 \cos(\omega t) \hat{k}$. Assume ideal wires, and neglect the self-inductance of the wires except for the inductor L .



[7 Pts.]

- (a) Find the induced emf in each loop.

The magnetic flux through each square loop is

$$\Phi_B(t) = B_0 a^2 \cos(\omega t).$$

Using Faraday's law,

$$\mathcal{E}(t) = -\frac{d\Phi_B}{dt} = \omega B_0 a^2 \sin(\omega t).$$

[9 Pts.]

- (b) Determine the amplitude of the current through the inductor.

Define the emf amplitude $\mathcal{E}_0 = \omega B_0 a^2$. Let the clockwise mesh currents in the left and right loops be $I_1(t)$, $I_3(t)$, and let the current through the middle resistor be $I_2(t)$. From the junction rule,

$$I_1 - I_2 = I_3,$$

Now apply Kirchhoff's loop rules,

$$I_1 + I_2 = \frac{\mathcal{E}(t)}{R}, \quad L \frac{dI_3}{dt} - R I_2 = \mathcal{E}(t).$$

Then,

$$I_1 = \frac{1}{2} I_3 + \frac{\mathcal{E}(t)}{2R}, \quad I_2 = -\frac{1}{2} I_3 + \frac{\mathcal{E}(t)}{2R}.$$

and,

$$L \frac{dI_3}{dt} + \frac{R}{2} I_3 = \frac{3}{2} \mathcal{E}(t).$$

Using phasors,

$$\frac{\omega^2 L^2}{R^2} I_{30} + \frac{1}{4} I_{30} = \frac{4\mathcal{E}_0}{9R^2} \Rightarrow I_{30} = \frac{3\mathcal{E}_0}{\sqrt{R^2 + 4\omega^2 L^2}} = \frac{3\omega B_0 a^2}{\sqrt{R^2 + 4\omega^2 L^2}}.$$

[9 Pts.] (c) Determine the average power dissipated in the system.

The average power dissipated in the resistors is

$$\overline{P} = \frac{1}{2}R \left(\overline{I_1^2} + \overline{I_2^2} \right).$$

Using

$$I_1^2 + I_2^2 = \frac{I_3^2}{2} + \frac{\mathcal{E}^2}{2R^2}, \quad \Rightarrow \quad \overline{P} = \frac{R}{4}I_{30}^2 + \frac{\mathcal{E}_0^2}{4R}$$

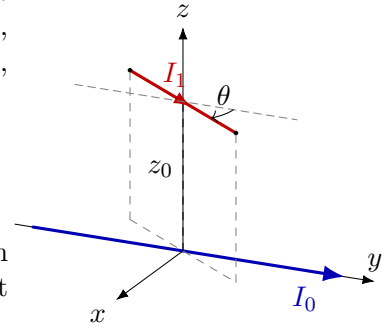
Therefore,

$$\overline{P} = \frac{\omega^2 B_0^2 a^4 (5R^2 + 2\omega^2 L^2)}{2R(R^2 + 4\omega^2 L^2)}.$$

2. An infinitely long straight wire lies along the y axis and carries a steady current I_0 in the $+y$ direction. A short straight wire segment of length ℓ carries a current I_1 . The segment lies in the plane $z = z_0$, so that its midpoint is located a distance z_0 from the infinite wire, and its endpoints are

$$\left(-\frac{\ell}{2} \sin \theta, -\frac{\ell}{2} \cos \theta, z_0\right), \quad \left(\frac{\ell}{2} \sin \theta, \frac{\ell}{2} \cos \theta, z_0\right).$$

Thus the segment makes an angle θ with the y axis, as shown in the figure. Neglect the magnetic field produced by the short segment itself.



Hint: $\int \frac{1}{1+x^2} dx = \arctan(x) + C$

- [5 Pts.] (a) Find the total magnetic force on the wire segment when $\theta = 0$.

$$\vec{F} = \frac{\mu_0 I_0 I_1}{2\pi z_0} \ell (-\hat{k}).$$

- [10 Pts.] (b) Find the torque on the wire segment about its midpoint when $\theta = 90^\circ$.

The magnetic field produced by the infinitely long wire is

$$\vec{B} = \frac{\mu_0 I_0}{2\pi(x^2 + z_0^2)} (z_0 \hat{i} - x \hat{k}).$$

For $\theta = 90^\circ$, the segment lies along the x -direction, so $d\vec{\ell} = dx \hat{i}$.

The force is

$$d\vec{F} = I_1 dx \hat{i} \times \vec{B} = \frac{\mu_0 I_0 I_1}{2\pi} \frac{x}{x^2 + z_0^2} dx \hat{j}.$$

The torque about the midpoint is

$$d\vec{\tau} = \vec{r} \times d\vec{F} = x \hat{i} \times d\vec{F}.$$

Therefore,

$$\vec{\tau} = \frac{\mu_0 I_0 I_1}{2\pi} \int_{-\ell/2}^{\ell/2} \frac{x^2}{x^2 + z_0^2} dx \hat{k}.$$

Using $\frac{x^2}{x^2 + z_0^2} = 1 - \frac{z_0^2}{x^2 + z_0^2}$, we get

$$\vec{\tau} = \frac{\mu_0 I_0 I_1}{2\pi} \left[\ell - 2z_0 \tan^{-1} \left(\frac{\ell}{2z_0} \right) \right] \hat{k}.$$

- [10 Pts.] (c) Find the total magnetic force on the wire segment for arbitrary angle θ .

The wire segment may be parametrized as $d\vec{\ell} = (\sin \theta \hat{i} + \cos \theta \hat{j}) ds$, and $x = s \sin \theta$. Then, we have

$$\vec{B} = \frac{\mu_0 I_0}{2\pi(z_0^2 + s^2 \sin^2 \theta)} (z_0 \hat{i} - s \sin \theta \hat{k}).$$

Then

$$d\vec{F} = \frac{\mu_0 I_0 I_1}{2\pi} \frac{(-s \sin \theta \cos \theta \hat{i} + s \sin^2 \theta \hat{j} - z_0 \cos \theta \hat{k})}{z_0^2 + s^2 \sin^2 \theta} ds.$$

The $\hat{i}$ and $\hat{j}$ components are odd functions of s and integrate to zero. Thus only the $\hat{k}$ component remains:

$$\vec{F} = -\frac{\mu_0 I_0 I_1 z_0 \cos \theta}{2\pi} \int_{-\ell/2}^{\ell/2} \frac{ds}{z_0^2 + s^2 \sin^2 \theta} \hat{k}.$$

Using

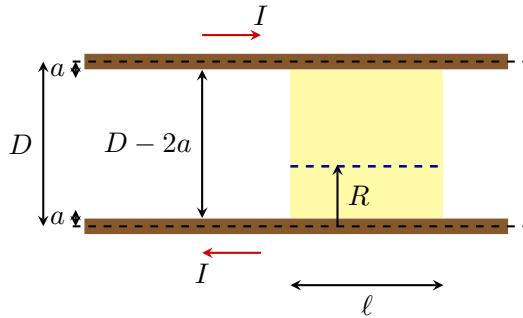
$$\int \frac{ds}{z_0^2 + s^2 \sin^2 \theta} = \frac{1}{z_0 \sin \theta} \tan^{-1} \left(\frac{s \sin \theta}{z_0} \right),$$

we obtain

$$\vec{F} = -\frac{\mu_0 I_0 I_1}{\pi} \cot \theta \tan^{-1} \left(\frac{\ell \sin \theta}{2z_0} \right) \hat{k}.$$

For $\theta \rightarrow 0$, this reduces to $\vec{F} = -\frac{\mu_0 I_0 I_1 \ell}{2\pi z_0} \hat{k}$, which agrees with part (a).

3. Two long straight parallel wires carry a current I , as shown in the figure. Each wire has a radius a and the axes are separated by distance D .



[8 Pts.]

- (a) Find the magnitude of the magnetic field at a point between the wires in the plane containing the axes as a function of R .

Let R be the distance from the lower wire to a point between the two wires. Then the distance from the upper wire is $D - R$. The magnetic fields due to the two wires are in the same direction in the region between the wires, so

$$B(R) = \frac{\mu_0 I}{2\pi R} + \frac{\mu_0 I}{2\pi(D - R)} = \frac{\mu_0 I}{2\pi} \frac{D}{R(D - R)}.$$

[10 Pts.]

- (b) In this plane, consider a surface of length ℓ and width $D - 2a$ between the wires, and determine the magnetic flux through the area $\ell(D - 2a)$.

The magnetic flux is

$$\Phi = \int_a^{D-a} B(R) \ell dR.$$

Substituting $B(R)$,

$$\Phi = \frac{\mu_0 I \ell}{2\pi} \int_a^{D-a} \left(\frac{1}{R} + \frac{1}{D - R} \right) dR = \frac{\mu_0 I \ell}{2\pi} \left(\ln \left(\frac{D - a}{a} \right) + \int_a^{D-a} \frac{dR}{D - R} \right).$$

Therefore,

$$\Phi = \frac{\mu_0 I \ell}{\pi} \ln \left(\frac{D - a}{a} \right).$$

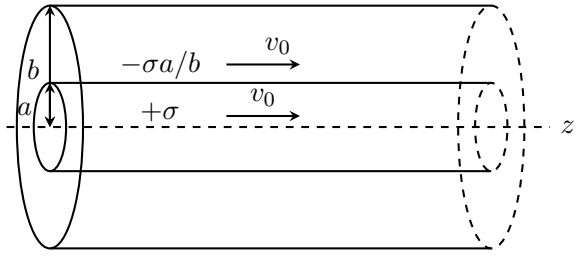
[7 Pts.]

- (c) Find the inductance per unit length for this two-conductor line. (Neglect the effect of the magnetic field inside the wires.)

The inductance is defined by $\Phi = LI$. Therefore the inductance per unit length is

$$\frac{L}{\ell} = \frac{\Phi}{I \ell} = \frac{\mu_0}{\pi} \ln \left(\frac{D - a}{a} \right).$$

4. Two infinitely long, thin coaxial cylindrical shells are centered on the z -axis. The inner shell has radius a , and the outer shell has radius b with $b > a$. The inner shell carries a uniform surface charge density $+\sigma$, while the outer shell carries a uniform surface charge density $-\sigma a/b$, so that the total charge per unit length of the system is zero. Both cylindrical shells move in the $+z$ direction with the same speed v_0 . Calculate the following quantities, giving both magnitude and direction when appropriate:



[4 Pts.]

- (a) the electric field for
- $a < r < b$
- ;

Using Gauss's law,

$$E(2\pi r\ell) = \frac{Q_{\text{enc}}}{\epsilon_0} = \frac{\sigma(2\pi a\ell)}{\epsilon_0}.$$

Thus,

$$E = \frac{\sigma a}{\epsilon_0 r} \quad \Rightarrow \quad \vec{E} = \frac{\sigma a}{\epsilon_0 r} \hat{r}, \quad a < r < b.$$

[4 Pts.]

- (b) the magnetic field for
- $a < r < b$
- ;

The inner positive shell produces a surface current density $J_{\text{in}} = \sigma v_0$. For $a < r < b$, an circle of radius r encloses only the current on the inner shell:

$$I_{\text{enc}} = J_{\text{in}}(2\pi a) = 2\pi a\sigma v_0.$$

Using Ampere's law,

$$B(2\pi r) = \mu_0 I_{\text{enc}} = \mu_0(2\pi a\sigma v_0) \quad \Rightarrow \quad B = \frac{\mu_0 \sigma a v_0}{r}.$$

By the right-hand rule, the magnetic field is in the $\hat{\phi}$ -direction.

[3 Pts.]

- (c) the electric field and the magnetic field for
- $r > b$

For $r > b$, the total charge per unit length is

$$\lambda_{\text{enc}} = 2\pi a\sigma + 2\pi b\left(-\frac{\sigma a}{b}\right) = 2\pi a\sigma - 2\pi a\sigma = 0 \quad \Rightarrow \quad \vec{E} = 0, \quad r > b.$$

The total current per unit length is also zero, since

$$I_{\text{enc}} = 2\pi a\sigma v_0 + 2\pi b\left(-\frac{\sigma a}{b}\right)v_0 = 0 \quad \Rightarrow \quad \vec{B} = 0, \quad r > b.$$

[4 Pts.]

- (d) the electromagnetic energy density for
- $a < r < b$
- ;

$$u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2.$$

So,

$$u = \frac{\sigma^2 a^2}{2\epsilon_0 r^2} + \frac{\mu_0 \sigma^2 a^2 v_0^2}{2r^2} = \frac{\sigma^2 a^2}{2\epsilon_0 r^2} \left(1 + \frac{v_0^2}{c^2}\right), \quad a < r < b.$$

where

$$c^2 = \frac{1}{\mu_0 \epsilon_0}.$$

- [4 Pts.] (e) the Poynting vector for $a < r < b$;

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{1}{\mu_0} EB \hat{k} = \frac{\sigma^2 a^2 v_0}{\varepsilon_0 r^2} \hat{k}, \quad a < r < b.$$

- [3 Pts.] (f) the total electromagnetic energy stored per unit length in the region between the cylinders;

$$\frac{U}{\ell} = \int_a^b u(2\pi r) dr = \frac{\pi \sigma^2 a^2}{\varepsilon_0} \left(1 + \frac{v_0^2}{c^2}\right) \int_a^b \frac{dr}{r} = \frac{\pi \sigma^2 a^2}{\varepsilon_0} \left(1 + \frac{v_0^2}{c^2}\right) \ln \left(\frac{b}{a}\right).$$

- [3 Pts.] (g) whether there exists a value of v_0 for which the electric and magnetic contributions to the energy density are equal everywhere in the region $a < r < b$, and if so, find it.

The electric and magnetic energy densities are equal when

$$\frac{1}{2} \varepsilon_0 E^2 = \frac{1}{2 \mu_0} B^2 \quad \Rightarrow \quad v_0^2 = \frac{1}{\mu_0 \varepsilon_0} = c^2.$$

Therefore,

$$v_0 = c.$$

Thus the electric and magnetic contributions to the energy density are equal everywhere between the cylinders only if the cylinders move at the speed of light.